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ON THE EQUIVALENCE OF μ-INVARIANT MEASURES FOR THE MINIMAL PROCESS AND ITS q-MATRIX

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In this paper we obtain necessary and sufficient conditions for a measure or vector that is μ-invariant for a q-matrix, $\boldsymbol{Q}$, to be μ-invariant for the family of transition matrices, $\{\boldsymbol{P}(t)\}$, of the minimal process it generates. Sufficient conditions are provided in the case when $\boldsymbol{Q}$ is regular and these are shown not to be necessary. When μ-invariant measures and vectors can be identified, they may be used, in certain cases, to determine quasistationary distributions for the process.

invariant measures * quasistationary distributions

1. Introduction

In [20] Tweedie established a relationship between μ-subinvariant measures and vectors for a standard irreducible q-matrix, $\boldsymbol{Q}$, and the family, $\{\boldsymbol{P}(t)\}$, of transition matrices of the corresponding minimal process, thus extending the classical results on subinvariance (the $\mu=0$ case) [11] (see also [14]). Earlier, Vere-Jones [23] proved that any μ-invariant measure for $\{\boldsymbol{P}(t)\}$ is also μ-invariant for $\boldsymbol{Q}$, but left open the general problem of determining μ-invariant quantities for $\{\boldsymbol{P}(t)\}$ from the q-matrix. The importance of identifying μ-invariant measures and vectors directly from the q-matrix is well recognised for seldom do we have at our disposal an expression for the transition probabilities. If we can identify them they may be used, in certain cases, to determine quasistationary distributions if the process terminates in a finite time (see, for example, [19, 3, 23, 2, 4, 5, 15 and 1]). Tweedie realised that in the irreducible λ-recurrent case, where λ is the decay parameter [13] of the process, any λ-invariant measure for $\boldsymbol{Q}$ is also λ-invariant for $\{\boldsymbol{P}(t)\}$, and, although he did not provide a relationship in the general case he did indicate that regularity of the q-matrix might be a key to the problem.

The main result of this paper is to provide necessary and sufficient conditions for μ-invariant measures and vectors for $\boldsymbol{Q}$ to be μ-invariant for $\{\boldsymbol{P}(t)\}$. The conditions can be expressed in terms of regularity of certain associated q-matrices but, as we shall see, not necessarily of the generator itself. Our method of proof is an elaboration of that used by Kelly [9] for invariant measures (the $\mu=0$ case) and can be traced back to Kendall's arguments [11] for symmetrically reversible processes. We open up the possibility of identifying quasistationary distributions for λ-transient processes directly from the q-matrix. Since we do not exclude the possibility that Q might not be regular our results also allow us to identify quasistationary distributions for processes that terminate by passing through infinitely many states in a finite time [20, 16].

In Section 2 we collect together various results on continuous-time Markov processes. Section 3 contains the main result together with some sufficient conditions for the existence of μ-invariant measures and vectors for the minimal process. In Section 4 we determine admissible values for μ giving particular attention to the case when the state space is reducible, while in Section 5 we show how the Harris-Veech condition for discrete-time processes can be used to determine necessary and sufficient conditions for the existence of measures and vectors that are μ-invariant for the q-matrix. A number of examples of Markov processes are provided to illustrate the results of the paper and in each case we identify all the μ-invariant measures and vectors.

2. Preliminaries

Let $Q=(q_{jk},\ j, k\in S)$ be a stable, conservative q-matrix over a countable state space S, that is, a collection of real numbers satisfying

$$0\leqslant q_{jk}<\infty, \quad k\neq j,$$

$$0\leqslant -q_{jj}\triangleq q_j<\infty, \quad j\in S,$$

and

$$\sum_{k\neq j} q_{jk}=q_j, \quad j\in S. \tag{1}$$

A standard time-homogeneous Markov process, $(X(t),\ t\geqslant 0)$, taking values in S can be constructed from Q using the method of Feller: The process starts in some initial state, $X(0)=j$, where it stays for a period exponentially distributed with parameter q_j and then moves to another state, k, with probability $(1-\delta_{jk})q_{jk}/q_j$ where δ_{jk} denotes the Kronecker delta; it stays in state k for a period that is exponentially distributed with parameter q_k, and so forth. The process may terminate by some finite time, T, at which the process is said to have exploded, since it has made infinitely many transitions in a finite time. Arbitrary rules for restarting the process after an explosion give rise to the possibility of infinitely many Markov processes on $[0,\infty)$ with the same q-matrix, Q. The (stationary) transition probabilities,

$$\pi_{jk}(t)=P\{X(t)=k\,|\,X(0)=j\}, \quad j, k\in S,$$

for each such process are differentiable for all $t>0$ and satisfy the backward equations,

$$\pi'_{jk}(t)=\sum_{i\in S} q_{ji}\pi_{ik}(t),\quad j,k\in S.$$

The minimal solution to these equations, that is, $P_{jk}(t)$ satisfying $P_{jk}(0)=\delta_{jk}$ and $P_{jk}(t)\le \pi_{jk}(t)$ for all $t>0$ where $\pi_{jk}(t)$ is any solution, is given by

$$P_{jk}(t)=P\{X(t)=k,\, t<T\,|\,X(0)=j\}$$

and it is straightforward to check that

$$P'_{jk}(0+)=q_{jk},\quad j,k\in S.$$

The process, $(X(t), 0\le t<T)$, with transition probabilities $\boldsymbol{P}(t)=(P_{jk}(t), j,k\in S)$ is the so-called *minimal process*. It is the essentially unique process determined by $\boldsymbol{Q}$ if the terminal time, T, is infinite with probability 1, whatever the initial starting state. In this case $\boldsymbol{Q}$ is said to be *regular*, or otherwise, *explosive*. Clearly, for all $t>0$,

$$P\{T>t\,|\,X(0)=j\}=\sum_{k\in S} P_{jk}(t)\le 1$$

and thus for $\boldsymbol{Q}$ to be regular it is necessary and sufficient that $\boldsymbol{P}(t)$ be stochastic for all $t>0$. It can be shown [18, Section 5.3] that this is equivalent to stipulating that equations

$$\sum_{k\in S} q_{jk}\xi_k=\alpha\xi_j,\quad j\in S,$$

possess no bounded, non-trivial, non-negative solution, $\boldsymbol{\xi}$, for some (and then for all) $\alpha>0$; an appropriate choice is $\alpha=1$. It is often difficult to verify the existence, or otherwise, of such a solution. However, S finite or $\{q_j,\, j\in S\}$ bounded is a sufficient condition for regularity.

The Feller construction of the minimal process shows that family $\{\boldsymbol{P}(t)\}_{t\ge 0}$ can be constructed from either the forward integral recurrence (FIR) or the backward integral recurrence (BIR). That is, $P_{jk}(t)$ can be realised as the limit of a non-decreasing sequence, $\{f_{jk}(n),\, n=0,1,2,\ldots\}$, where

$$f_{jk}(t,0)=\delta_{jk}\,\mathrm{e}^{-q_jt} \tag{2}$$

and either

$$f_{jk}(t,n+1)=f_{jk}(t,0)+\sum_{i\ne j}\int_0^t q_{ji}f_{ik}(u,n)\,\mathrm{e}^{-q_j(t-u)}\,\mathrm{d}u\quad \text{(BIR)},$$

or

$$f_{jk}(t,n+1)=f_{jk}(t,0)+\sum_{i\ne k}\int_0^t f_{ji}(u,n)q_{ik}\,\mathrm{e}^{-q_k(t-u)}\,\mathrm{d}u\quad \text{(FIR)}.$$

The quantity $f_{jk}(t,n)$ represents the probability that the process is in state k after at most n transitions given that it started in state j.

The q-matrix is said to be *irreducible* if for each pair of states, j and k, $P_{jk}(t)$ is positive for some (and then for all) $t>0$; this is equivalent to the more useful condition that for all pairs of states, j and k, there exists a finite sequence of states, $i_1, i_2, \ldots, i_r$, distinct from one another and from j and k, such that

$$q_{j,i_1} q_{i_1,i_2} \cdots q_{i_r,k} > 0.$$

That this condition is sufficient is essentially immediate but the proof of its necessity is not obvious (see [11, pp. 425–6]). If it fails $\boldsymbol{Q}$ is reducible in that S can be partitioned into classes such that over each class, C, the submatrix $\boldsymbol{Q}_C = (q_{jk}, j, k \in C)$ is irreducible.

Corresponding to each irreducible class, C, there exists a finite non-negative constant $\lambda = \lambda(C)$, called the *decay parameter* of that class, such that, for all j and k in C,

$$t^{-1} \log P_{jk}(t) \to -\lambda \tag{3}$$

as $t \to \infty$ (Kingman [13]).

Following Kingman we shall call a collection of positive numbers $\boldsymbol{m} = (m_j, j \in C)$ a *μ-subinvariant measure on C ($\subseteq S$) for the family* $\{\boldsymbol{P}(t)\}$ if, for all k in C and $t>0$,

$$\sum_{j \in C} m_j P_{jk}(t) \leq \mathrm{e}^{-\mu t} m_k$$

and *μ-invariant* if equality holds; if $\mu = 0$ these correspond to the more familiar invariance notions. In contrast we shall call a collection of positive quantities $\boldsymbol{x} = (x_j, j \in C)$ a *μ-subinvariant vector on C for* $\{\boldsymbol{P}(t)\}$ if, for all j in C and $t>0$,

$$\sum_{k \in C} P_{jk}(t) x_k \leq \mathrm{e}^{-\mu t} x_j$$

and *μ-invariant* with equality. Henceforth we shall restrict our attention to either one of two cases, namely when C is the whole state space, S, or, any irreducible class. In the latter case one possible value for μ is the decay parameter, $\lambda(C)$, and Theorem 4 of [23] shows that for a μ-invariant measure to exist it is necessary that $0 \leq \mu \leq \lambda(C)$. If $C = S$ and is reducible we shall see it is necessary that μ does not exceed any of the decay parameters.

Given any μ-subinvariant measure, $\boldsymbol{m}$, and vector, $\boldsymbol{x}$, over C we can define two substochastic families $\{\boldsymbol{P}^*(t)\}$ and $\{\bar{\boldsymbol{P}}(t)\}$ of transition matrices over C by

$$P^*_{jk}(t) = \mathrm{e}^{\mu t} m_k P_{kj}(t)/m_j, \quad j, k \in C,$$

and

$$\bar{P}_{jk}(t) = \mathrm{e}^{\mu t} P_{jk}(t) x_k / x_j, \quad j, k \in C.$$

If C is irreducible we can append to C an absorbing state, δ, so that $\boldsymbol{P}^*$ and $\bar{\boldsymbol{P}}$ are stochastic over $C^+ = C \cup \{\delta\}$ and identify C as being either transient, or positive or null-recurrent. It is precisely corresponding to these categories in the $\mu = \lambda(C)$ case that Kingman [13], following the nomenclature of Vere-Jones [22], identified the notions of λ-transience, and λ-positive or null-recurrence. Thus C is said to be *λ-transient* or *λ-recurrent* according as

$$\int_0^\infty P_{jk}(t)\,\mathrm{e}^{\lambda t}\,\mathrm{d}t$$

converges or diverges for some (and then for all) j and k in C and in the latter case as being *λ-positive* or *λ-null* according as

$$\lim_{t\to\infty} P_{jk}(t)\,\mathrm{e}^{\lambda t}$$

is positive or zero for some (and then for all) j and k in C. Observe that if C is recurrent λ must be zero and so these notions are non-trivial only when C is transient.

The matrix $\boldsymbol{P}^*$ is often referred to in the literature as the "reverse" or even "time-reverse" transition matrix. However in the light of the recent refinements of the notion of reversibility this nomenclature seems misleading. Observe that if $\boldsymbol{m}$ and $\boldsymbol{x}$ are *any* μ-subinvariant measure and vector on C then, for all $t \geqslant 0$,

$$u_j P^*_{jk}(t) = u_k \bar{P}_{kj}(t), \quad j, k \in C, \tag{4}$$

where $u_j = m_j x_j$. That is, $\boldsymbol{P}^*$ is the *time-reverse of $\bar{\boldsymbol{P}}$ with respect to $u = (u_j, j \in C)$* [9]. Indeed summing over k (respectively j) in (4) shows that $\boldsymbol{u}$ is invariant for $\bar{\boldsymbol{P}}$ (respectively $\boldsymbol{P}^*$) if and only if $\boldsymbol{P}^*$ (respectively $\bar{\boldsymbol{P}}$) is stochastic.

Since there is a one-to-one correspondence between the q-matrix $\boldsymbol{Q}$ and the family $\{\boldsymbol{P}(t)\}$, it should not be surprising that μ-invariant measures and vectors can be determined from $\boldsymbol{Q}$. Call $\boldsymbol{m} = (m_j, j \in C)$, where $C \subseteq S$, a *μ-subinvariant measure on C for $\boldsymbol{Q}$* if it has positive entries that satisfy

$$\sum_{j\in C} m_j q_{jk} \leqslant -\mu m_k, \quad k \in C, \tag{5}$$

for all k in C and *μ-invariant* if equality obtains. The analogous notions of μ-subinvariance and invariance of a *vector*, $\boldsymbol{x}$, are defined in terms of the inequalities

$$\sum_{k\in C} q_{jk} x_k \leqslant -\mu x_j \qquad j \in C. \tag{6}$$

Tweedie proves [20, Propositions 1 and 2] the following result.

Theorem 1. (i) *If $\boldsymbol{m}$ is a μ-subinvariant measure (vector) on C for $\boldsymbol{Q}$, with $\mu \leqslant \inf_{j\in C} q_j$, then it is μ-subinvariant on C for $\{\boldsymbol{P}(t)\}$.*

(ii) *Conversely, if $\boldsymbol{m}$ is a μ-subinvariant measure (vector) on C for $\{\boldsymbol{P}(t)\}$ it is also μ-subinvariant on C for $\boldsymbol{Q}$ and $\mu \leqslant \inf_{j\in C} q_j$.*

(iii) *If $\boldsymbol{m}$ is a μ-invariant measure (vector) on C for $\{\boldsymbol{P}(t)\}$ it is μ-invariant on C for $\boldsymbol{Q}$. If C is λ-recurrent and $\mu = \lambda(C)$ the converse is true.*

These assertions are proved for the case when C is the whole state space, S, assumed to be irreducible, and Tweedie rightly indicates this assumption is not essential. Indeed the result holds good if C is any irreducible class and can be extended to the reducible case. However, as we shall see, care must be taken in determining a range of admissible values for μ.

Associated with the minimal process there is a discrete-time Markov process, $(X_n,\ n=0,1,2,\dots)$ called the *jump-chain* which records the sequence of states visited by the process. Its transition matrix, $\boldsymbol{J}$, has elements given by

$$\begin{aligned} J_{jk} &= \delta_{jk} & &\text{if } q_j = 0, \\ &= (1-\delta_{jk})q_{jk}/q_j & &\text{if } q_j > 0. \end{aligned}$$

Many properties of the jump-chain can be related to properties of the process. We shall use the fact that they share the same irreducible classes and that provided S contains no absorbing states, $\boldsymbol{u}=(u_j, j\in S)$ is an invariant measure (respectively vector) on S for $\boldsymbol{Q}$ if and only if $\boldsymbol{v}=(u_jq_j,\ j\in S)$ (respectively $\boldsymbol{u}$) is an invariant measure (respectively vector) on S for $\boldsymbol{J}$(see, for example, [8, Exercise 1.1.5]); this result holds good whether or not $\boldsymbol{Q}$ is conservative.

3. μ-Invariance and the q-matrix

In this section we examine the relationship between μ-invariant measures and vectors for the q-matrix and those for the corresponding family of transition matrices for the minimal process it generates. Before stating the main result let us observe that for $\boldsymbol{m}$ to be a μ-subinvariant measure (or vector) it is necessary that $\mu \leqslant \inf_{j\in C} q_j$, viz.

$$0 \leqslant \sum_{j\neq k} m_j q_{jk} = \sum_{j\in C} m_j q_{jk} + m_k q_k \leqslant (q_k - \mu) m_k. \tag{7}$$

Theorem 2. *Let $\boldsymbol{Q}$ be a stable conservative q-matrix over a countable state space S and let $\{\boldsymbol{P}(t)\}$ be the family of transition matrices for the minimal process generated by $\boldsymbol{Q}$. Let C be either the whole state space or any irreducible class thereof.*

If $\boldsymbol{m}$ is a μ-invariant measure on C for $\boldsymbol{Q}$, then it is μ-invariant on C for $\{\boldsymbol{P}(t)\}$ if and only if the equations

$$\sum_{j\in C} y_j q_{jk} = -\nu y_k, \quad 0 \leqslant y_k \leqslant m_k, \ k\in C, \tag{8}$$

have no non-trivial solution for some (and then for all) $\nu<\mu$.

If $\boldsymbol{x}$ is a μ-invariant vector on C for $\boldsymbol{Q}$, then it is μ-invariant on C for $\{\boldsymbol{P}(t)\}$ if and only if the equations

$$\sum_{k\in C} q_{jk} z_k = -\nu z_j, \quad 0 \leqslant z_j \leqslant x_j, \ j\in C, \tag{9}$$

have no non-trivial solution for some (and then for all) $\nu<\mu$.

Proof. (i) If $\boldsymbol{m}$ satisfies (5) with equality and $\mu \leqslant \inf_{j\in C} q_j$ then $\boldsymbol{Q}^* = (q^*_{jk}, j, k \in C)$ defined by

$$q^*_{jk} = (q_{kj} + \mu\delta_{jk}) m_k / m_j, \quad j, k \in C,$$

is a stable conservative q-matrix over C. Define $f^*_{jk}(t, n)$ by (2) and (FIR) in terms of $\boldsymbol{Q}^*$ and let

$$P^*_{jk}(t) = \lim_{n\to\infty} f^*_{jk}(t, n), \quad j, k \in C.$$

We shall show by induction that, for $n = 0, 1, \ldots,$

$$m_j f_{jk}(t, n) = \mathrm{e}^{-\mu t} m_k f^*_{kj}(t, n), \quad j, k \in C, \tag{10}$$

where $\{f_{jk}(n),\ n = 0, 1, \ldots\}$ is the non-decreasing sequence, with limit $P_{jk}(t)$, that satisfies (2) and (BIR). Clearly

$$m_j f_{jk}(t, 0) = \mathrm{e}^{-\mu t} m_k f^*_{kj}(t, 0), \quad j, k \in C, \tag{11}$$

since $q_j = q^*_j + \mu$. Now assume (10) is true for some $n \geqslant 0$. From (BIR), for all $j, k \in C$, we have

$$m_j f_{jk}(t, n+1) = m_j f_{jk}(t, 0) + \sum_{\substack{i\in S\\ i\neq j}} \int_0^t m_j q_{ji} f_{ik}(u, n)\, \mathrm{e}^{-q_j(t-u)}\, \mathrm{d}u. \tag{12}$$

If C is not the whole of S but rather any irreducible class, there is no contribution to the sum for $i \notin C$ since for each such i either $q_{ji} = 0$ or $f_{ik}(u, n) = 0$ for all $n \geqslant 0$ and $t \geqslant 0$. But, by the definition of $\boldsymbol{Q}^*$, we have $m_j q_{ji} = m_i q^*_{ij}$ for all $i \neq j$ which, together with the inductive hypothesis, shows that

$$m_j q_{ji} f_{ik}(u, n) = m_k f^*_{ki}(u, n) q^*_{ij}\, \mathrm{e}^{-\mu t}.$$

Substituting this in (12) and using (11) together with the (FIR) definition of $f^*_{kj}(t, n+1)$ shows that

$$m_j f_{jk}(t, n+1) = \mathrm{e}^{-\mu t} m_k f^*_{kj}(t, n+1)$$

for all j and k in C, thus completing the induction. Taking limits as $n \to \infty$ we arrive at

$$m_j P_{jk}(t) = \mathrm{e}^{-\mu t} m_k P^*_{kj}(t). \tag{13}$$

On summing this expression over j in C we see that $\boldsymbol{m}$ is μ-invariant for $\{P(t)\}$ if and only if

$$\sum_{j\in C} P^*_{kj}(t) = 1$$

for all k in C and $t > 0$. This is equivalent to stipulating that $\boldsymbol{Q}^*$ be regular, and a necessary and sufficient condition for this is that the equations

$$\textstyle\sum_{k\in C} q^*_{jk}\xi_k = \alpha\xi_j, \quad 0 \leqslant \xi_j \leqslant 1,\ j \in C,$$

have no non-trivial solution for some (and then for all) $\alpha > 0$. But, by the definition of Q^*, these equations can be written

$$\sum_{k\in C} m_k \xi_k q_{kj} = (\alpha - \mu) m_j \xi_j, \quad j \in C.$$

Setting $y_j = m_j \xi_j$ and $\nu = \mu - \alpha$ achieves the desired result.

(ii) If x satisfies (6) with equality the proof is almost exactly the same. We define $\bar{Q}$, a stable conservative q-matrix over C, by

$$\bar{q}_{jk} = (q_{jk} + \mu\delta_{jk}) x_k / x_j, \quad j, k \in C, \tag{14}$$

and then show that

$$P_{jk}(t) x_k = e^{-\mu t} x_j \bar{P}_{jk}(t), \quad j, k \in C,$$

where $\bar{P}_{jk}(t)$, $j, k \in C$ are the transition probabilities generated by $\bar{Q}$. This leads us to the requirement that $\bar{Q}$ be regular and so to the desired necessary and sufficient conditions for μ-invariance of x for $\{P(t)\}$.

Remark. The assumption that Q is conservative is only made for convenience. If the equality in (1) is replaced by an inequality the conclusions of the theorem do not alter, only that the appropriate extended definition of the family $\{P(t)\}$ [6, Section 5.6] should be used.

If m is a μ-invariant measure (or vector) on C for Q there are a number of conditions that are sufficient for m to be invariant for $\{P(t)\}$. Clearly C finite or otherwise $\{q_j, j \in C\}$ bounded is sufficient, since then both $\{q_j^*, j \in C\}$ and $\{\bar{q}_j, j \in C\}$ are bounded implying that each of Q^* and $\bar{Q}$ is regular. It is clear also that Tweedie's [20] condition that C be irreducible and λ-recurrent and $\mu = \lambda$ can be obtained immediately from Theorem 2 since then, from (13) and the equivalent expression involving x and $\{P(t)\}$, both $\int_0^\infty P_{jk}^*(t)\,dt$ and $\int_0^\infty \bar{P}_{jk}(t)\,dt$ diverge and the ensuing recurrence of Q^* and $\bar{Q}$ imply their regularity. Another sufficient condition is provided by Vere-Jones in [23], where attention is restricted to the important special case of when $S = C \cup \{0\}$ and 0 is an absorbing state. Vere-Jones proved that if Q is conservative and regular then any μ-invariant measure, m, on C for Q such that $\sum_{j\in C} m_j < \infty$ is also μ-invariant for $\{P(t)\}$. To see how these conditions arise in the context of Theorem 2 we refer to expression (13). On summing over $k \in C$ and using the fact that $\{P(t)\}$ is stochastic we arrive at

$$m_j(1 - P_{j0}(t)) = e^{-\mu t} \sum_{k\in C} m_k P_{kj}^*(t), \quad j \in C.$$

Now we refer to the proof of Theorem 6 of [23] from which it can easily be deduced that if Q is conservative and regular, $q_0 = 0$ and $\sum_{j\in C} m_j < \infty$ then

$$\sum_{j\in C} m_j P_{j0}(t) = (1 - e^{-\mu t}) \sum_{j\in C} m_j. \tag{15}$$

This combines with the previous expression to yield

$$\sum_{k\in C} m_k \left(1 - \sum_{j\in C} P_{kj}^*(t)\right) = 0.$$

Since the sum consists of non-negative terms it follows that $P^*(t)$ is stochastic (Q^* regular) and so the necessary and sufficient condition of the theorem is satisfied. The result can be extended in a number of ways. For example if C is the whole state, S, then from (13) we have that

$$m_j = \mathrm{e}^{-\mu t} \sum_{k \in S} m_k P^*_{kj}(t), \quad j \in S,$$

if Q is regular, and so, provided $\sum_{j \in S} m_j$ converges,

$$\sum_{k \in S} m_k \left(1 - \mathrm{e}^{-\mu t} \sum_{j \in S} P^*_{kj}(t) \right) = 0.$$

Again since the terms in the sum are non-negative it follows not only that Q^* is regular but that μ is of necessity zero. Thus the interesting case of $\mu > 0$ and $\sum_{j \in C} m_j < \infty$ can only occur when $C \neq S$. If S is assumed irreducible then the (0−) invariance of m can be established more directly, for if $\sum_{j \in S} m_j < \infty$ then Q is positive recurrent and m is the essentially unique invariant measure for $\{P(t)\}$ [12, 9]. To obtain a similar sufficient condition relating to μ-invariant vectors let us consider Reuter's condition for the regularity of Q in the two cases just considered. In either case this stipulates that if Q is regular any non-trivial, non-negative solution to equations (9) for say $\nu(=-\alpha) = -1$ is unbounded (note that in the case when 0 is an absorbing state $z_0 = 0$). Therefore if $\{x_j\}$ is bounded over C such a solution cannot be bounded above by x and so $\bar{Q}$ is regular. We have thus proved the following result.

Corollary 1. *Let Q be a stable, conservative and regular q-matrix over a countable state space S and let $\{P(t)\}$ be the (stochastic) family of transition matrices for the minimal process generated by Q. If C is the whole of S or such that $S = C \cup \{0\}$ where 0 is an absorbing state then the following statements are true.*

If m is a μ-invariant measure on C for Q then for m to be μ-invariant for $\{P(t)\}$ it is sufficient that $\sum_{j \in C} m_j$ converges.

If x is a μ-invariant vector on C for Q then a sufficient condition for x to be μ-invariant for $\{P(t)\}$ is that $\{x_j\}$ be bounded.

Remarks. (i) The conditions of the corollary appear in connection with quasi-stationary distributions (see, for example, Flashpohler [4]). If C is an irreducible class and $\sum_{j \in C} m_j < \infty$ then m, when properly normalized, has a quasistationary interpretation. If, in addition, $\{x_j\}$ is bounded then the measure $u = (m_j x_j, j \in C)$ also admits such an interpretation.

(ii) In deriving the sufficient condition for μ-invariance of m we arrived at the invariance condition of the theorem indirectly by referring to equations (15), involving probabilities $P_{j0}(t)$, yet we were able to derive a sufficient condition more directly when dealing with μ-invariant vectors owing to the similarity of the invariance condition to Reuter's condition for the regularity of Q. The invariance condition for m also bears a striking resemblance to another condition of Reuter [18, Section 6], namely that which ensures the existence of a unique solution to the forward equations

$$\pi'_{jk}(t) = \sum_{i \in S} \pi_{ji}(t) q_{ik}, \quad j, k \in S;$$

$\{P(t)\}$ is the only solution if and only if Q is regular *or* Q is explosive and there is no non-trivial, non-negative solution to

$$\sum_{j \in S} y_j q_{jk} = \alpha y_k, \quad k \in S,$$

such that $\sum_{j \in S} y_j < \infty$ for some (and then for all) $\alpha > 0$. This similarity can be exploited in an analogous way, for if the conditions of Corollary 1 are satisfied and the forward equations have a unique solution, then any non-trivial, non-negative solution to (8) for say $\nu\ (= -\alpha) = -1$ is such that $\sum_{j \in C} y_j$ diverges. (Note that in the case when $S = C \cup \{0\}$ y_0 does not appear in the equations for $k \in C$, but is determined by $(y_k, k \in C)$.) If $\sum_{j \in C} m_j < \infty$ then such a solution cannot be bounded above by m and so Q^* is regular. However observe that although this argument is relatively straightforward the conclusions of Corollary 1 remain valid whether or not the solution to the forward equations is unique.

(iii) In the corollary we assume that Q is regular. It should be emphasised, however, that Theorem 2 is valid even when this is not the case. In our first example we demonstrate, among other points raised in this section, that it is possible for Q to be explosive yet $\{P(t)\}$ might have a bounded μ-invariant measure, m. If, in such a situation, the process is certain to terminate m has a quasistationary interpretation provided $\sum_{j \in C} m_j$ converges [20, 16].

Example 1. We shall consider a simple random walk on the integers. For each j in $\mathbb{Z}$ let $q_{j,j+1} = -q_{jj} = q_j > 0$, with all other transition rates zero. This defines a stable conservative q-matrix, Q, which is regular if and only if

$$\sum_{j=0}^{\infty} q_j^{-1} = \infty$$

(see, for example, [9]), and one for which each state is in a class by itself. We shall establish the existence of μ-invariant measures and vectors for Q and determine whether or not they are μ-invariant for $\{P(t)\}$. For every non-negative μ strictly less than any of the $\{q_j\}$ the essentially unique non-trivial non-negative solutions to the equations (6) and (5) are, respectively,

$$x_j = \prod_{r=0}^{j-1} (1 - \mu q_r^{-1}), \qquad j > 0,$$

$$= \prod_{r=1}^{-j} (1 - \mu q_{-r}^{-1})^{-1}, \qquad j < 0,$$

and

$$m_j = (q_j - \mu)^{-1} x_j^{-1}, \quad j \in \mathbb{Z}.$$

The solution $(z_j, j \in C)$ to (9) and the corresponding solution $(y_j, j \in C)$ to (8) are clearly also related in this way. Thus setting $\xi_j = y_j/m_j$ and $\phi_j = z_j/x_j$ we have that $\xi_j = \phi_j^{-1}$, $j \in \mathbb{Z}$, where

$$\phi_j = \prod_{r=0}^{j-1}\left(1+\frac{(\mu-\nu)}{(q_r-\mu)}\right), \qquad j>0,$$

$$= \prod_{r=1}^{-j}\left(1-\frac{(\mu-\nu)}{(q_{-r}-\nu)}\right), \qquad j<0.$$

Observe that, for all $r \in \mathbb{Z}$,

$$0<\frac{\mu-\nu}{q_r-\nu}<1 \quad \text{and} \quad \frac{\mu-\nu}{q_r-\mu}>0$$

and so $\{\phi_j\}$ is unbounded if and only if

$$\sum_{j=0}^{\infty} q_j^{-1} = \infty, \tag{16}$$

or, equivalently, $\boldsymbol{Q}$ is regular, while $\{\xi_j\}$ is unbounded if and only if

$$\sum_{j=0}^{\infty} q_{-j}^{-1} = \infty. \tag{17}$$

Observe also that since $0 \leq \mu q_r^{-1} < 1$ condition (17) is necessary and sufficient for $\boldsymbol{x}$ to be unbounded, while (16) is necessary and sufficient for $\boldsymbol{m}$ to be unbounded, and so it is impossible for $\boldsymbol{m}$ to be bounded when $\boldsymbol{Q}$ is regular.

If conditions (16) and (17) both hold $\boldsymbol{Q}$ is regular and although $\boldsymbol{m}$ and $\boldsymbol{x}$ are μ-invariant for $\{P(t)\}$ neither is bounded, demonstrating that the conditions of Corollary 1 are not necessary. In contrast, if both conditions fail $\boldsymbol{Q}$ is not regular and although both $\boldsymbol{m}$ and $\boldsymbol{x}$ are bounded neither is μ-invariant for $\{\boldsymbol{P}(t)\}$. If (16) holds and (17) fails $\boldsymbol{Q}$ is regular, $\boldsymbol{x}$ is bounded and μ-invariant for $\{\boldsymbol{P}(t)\}$, while $\boldsymbol{m}$ is neither bounded nor μ-invariant for $\{\boldsymbol{P}(t)\}$. This demonstrates Corollary 1. The final combination provides a q-matrix that is not regular, a bounded μ-invariant measure for $\{\boldsymbol{P}(t)\}$ and no μ-invariant vector.

4. Permissible values for μ

We have already observed that if C is irreducible then for $\boldsymbol{m}$ (or indeed $\boldsymbol{x}$) to be μ-invariant for $\{\boldsymbol{P}(t)\}$ it is necessary that μ does not exceed the decay parameter of that class. The proof of Theorem 2 allows us to strengthen and extend this statement.

Theorem 3. *Let $\boldsymbol{Q}$ be a stable conservative q-matrix over a countable state space S, not necessarily irreducible. For $\boldsymbol{m}$ to be a μ-subinvariant measure (or vector) on S for*

Q it is necessary that μ does not exceed the decay parameter of any irreducible subclass, that is,

$$\mu \leqslant \inf_C \lambda(C).$$

Proof. Let $\boldsymbol{m}$ be any μ-subinvariant measure on S for Q. Modify the definition of Q^* given in the proof of Theorem 2 by appending to S a state, δ (an absorbing state), letting $q^*_{\delta\delta}=0$, $q^*_{\delta k}=0$ for each k in S and

$$q^*_{j\delta} = -\sum_{k\in S} q^*_{jk}$$

for each j in S. Clearly Q^* is a stable conservative q-matrix over the enlarged state space $S^+ = S \cup \{\delta\}$. Now define $P^*_{jk}(t)$, $j, k \in S$ as in the proof of Theorem 2. Since δ is an absorbing state there is no contribution to the sum in (12) for $i=\delta$, and thus following the same inductive argument we have

$$m_j P_{jk}(t) = \mathrm{e}^{-\mu t} m_k P^*_{kj}(t) \tag{18}$$

for each j and k in S; of course $P^*_{jk}(t)$ is defined when j or k equals δ but its value will not concern us.

Suppose now that j and k belong to the same irreducible class, C. We have, from (3), that

$$\lim_{t\to\infty} t^{-1}\log P_{jk}(t) = -\mu + \lim_{t\to\infty} t^{-1}\log P^*_{jk}(t)$$

and so, in an obvious notation, $\mu = \lambda(C) - \lambda^*(C) \leqslant \lambda(C)$. If $\boldsymbol{m}$ is a μ-subinvariant vector on S for Q then we arrive at the same conclusion if we employ the q-matrix $\bar{Q}$ and the family $\{\bar{P}(t)\}$. The proof is completed by observing that C is arbitrary.

Remark. We note in passing that in proving Theorem 3 we have almost provided a proof for the extension of Theorem 1(iii) to the case when S is reducible. If $\boldsymbol{m}$ is a μ-invariant measure on S for $\{P(t)\}$ then clearly $\boldsymbol{m}$ is μ-subinvariant for Q. Summing over $j \in S$ in (18) shows that P^* is strictly stochastic over S and therefore $P^*_{j\delta}(t) = 0$ for all j in S and $t \geqslant 0$. Since $(f^*_{j\delta}(t, n),\ n = 0, 1, \ldots)$ is non-decreasing with limit $P^*_{j\delta}(t)$ we have that $f^*_{j\delta}(t, n) = 0$ for all $n \geqslant 0$. In particular the (FIR) for $n=1$ shows that $q^*_{j\delta} = 0$ for all j in S. Thus Q^* is conservative over S which is equivalent to saying that $\boldsymbol{m}$ is strictly μ-invariant for Q over S. We remark also that if Q is regular and $\boldsymbol{m}$ is a μ-subinvariant *measure* on S for Q such that $\sum_{j\in S} m_j < \infty$ then it is necessary that $\mu = 0$, and this is consistent with earlier remarks.

A reader familiar with Kingman's early paper [13] will be aware that transition probabilities *between* classes decay more slowly than do those within classes, although it is not necessarily true that $t^{-1}\log P_{jk}(t)$ tends to a limit as $t \to \infty$. Weaker statements do exist, however, and one might suspect that they could, in some cases, entail further restrictions on μ. Suppose that $j \in C$ and $k \in C'$ where C and C' are two irreducible classes such that C' is accessible from C (sometimes written $C < C'$).

Then as $t \to \infty$ the limit infimum, $\underline{\lambda} = \underline{\lambda}(C, C')$, and the limit supremum, $\bar{\lambda} = \bar{\lambda}(C, C')$ of $-t^{-1} \log P_{jk}(t)$ exist and satisfy

$$0 \leq \underline{\lambda} \leq \bar{\lambda} \leq \min\{\lambda(C), \lambda(C')\}. \tag{19}$$

Thus taking limits in (18) we have, in an obvious notation, $\mu = \bar{\lambda} - \bar{\lambda}^* = \underline{\lambda} - \underline{\lambda}^*$. Therefore

$$\mu \leq \underline{\lambda} \leq \bar{\lambda} \leq \min\{\lambda(C), \lambda(C')\}$$

indicating a possibly tighter restraint on μ, namely that μ be allowed not to exceed

$$\inf_{C,C'} \underline{\lambda}(C, C')$$

where the infimum is taken over all pairs, (C, C'), of irreducible subclasses of S such that $C < C'$. This is of course more of a restriction but in most cases $\underline{\lambda}(C, C') = \bar{\lambda}(C, C') = \min\{\lambda(C), \lambda(C')\}$ and if not $\underline{\lambda}(C, C')$ is seldom at our disposal. In Example 1 above each state is in a class on its own and clearly $\{j\} < \{k\}$ if and only if $k \geq j$. Since the only possible transitions are of the form $j \to j+1$ we have that $P_{jj}(t) = e^{-q_j t}$ and so $\lambda(\{j\}) = q_j$. Further it is not difficult to show that if $k \geq j$, $\underline{\lambda}(\{j\}, \{k\}) = \bar{\lambda}(\{j\}, \{k\}) = \min(q_j, q_k)$. Thus it is necessary only that μ does not exceed $\inf_{j \in \mathbb{Z}} q_j$.

In our next example the only admissible value for μ is 0 yet $0 < \inf \lambda(C) = \inf q_j$. We shall consider a Markov process where transitions occur at points of a Poisson process and where the zero state has a prescribed first-recurrence-time distribution.

Example 2. Take the state space, S, to be the integers and define $\boldsymbol{Q}$ by

$$\begin{aligned} &q_{j,j-1} = -q_{jj} = 1, && j = 0, -1, -2, \ldots, \\ &q_{j0} = 1 - q_{j,j+1} = c_j/(1 - s_{j-1}), && \multirow{2}{*}{} j = 1, 2, \ldots, \\ &q_{jj} = -1, && \end{aligned}$$

where $c_0 = 0$ and $c_1, c_2, \ldots$ are non-negative constants such that

$$s_j = \sum_{r=0}^{j} c_r, \quad j = 0, 1, 2, \ldots,$$

is strictly less than 1. Observe that each state forms a class by itself and if the process starts in state $C = \{1\}$ it enters the state $C' = \{0\}$ either on its next jump or after traversing a path $1, 2, \ldots, n$ for some $n \geq 2$. The sequence $\{c_r\}$ has been chosen so that

$$P_{10}(t) = e^{-t} \sum_{r=1}^{\infty} c_r t^r / r!$$

If we set

$$\begin{aligned} c_r &= 1/k(k+1) \quad \text{when } r = n_k, \ k = 1, 2, \ldots, \\ &= 0 \qquad\qquad \text{otherwise,} \end{aligned}$$

for some strictly increasing sequence, $\{n_k\}$, of positive integers then it can be shown that $\underline{\lambda}(C, C') = 0$ [13], and so for μ-invariant measures and vectors on S to exist it is necessary that $\mu = 0$ independently of whether Q is regular; observe that $\lambda(\{j\}) = q_j = 1$ for all j. It is of interest to note also that the choice $n_k = 2^k$ leads to strict inequalities ($\underline{\lambda} < \bar{\lambda} < 1$) in (19) (see [13]).

We shall now attempt to find the invariant measures and vectors on S for Q. On writing down the equations (5) for μ-invariant measures we see that no choice of $\{c_r\}$ admits a non-trivial solution. The equations (6) for μ-invariant vectors, however, do admit a solution and although it is not difficult to write it down in the general case the form is rather cumbersome. The choice $c_r = 1/r(r+1)$, $r = 1, 2, \ldots$, ($n_k = k$ above) leads to a solution of the form

$$\begin{aligned} x_j &= j(1-\mu)^{j-1}\left(x_1 - x_0 \sum_{r=1}^{j-1} \frac{(1-\mu)^{-r}}{r(r+1)}\right), \quad && j > 0, \\ &= (1-\mu)^{-j}, && j < 0. \end{aligned} \tag{20}$$

Of course we know already that μ must of necessity be zero but we will need to consider equations (9) in checking (0−) invariance for $\{P(t)\}$ and these of course have, in essence, the same solution. We observe in passing that if $0 < \mu < 1$ the series in (20) diverges and so it is impossible to guarantee $x_j > 0$ for all j. When $\mu = 0$ we obtain (infinitely many) solutions of the form

$$\begin{aligned} x_j &= 1 + aj, \quad && j = 0, 1, 2, \ldots, \\ &= 1, && j = -1, -2, \ldots \end{aligned} \tag{21}$$

where $a \geq 0$ is arbitrary. The system of equations (9) have non-trivial solutions of the form

$$\begin{aligned} z_j &= j(1-\nu)^{j-1}\left(z_1 - z_0 \sum_{r=1}^{j-1} \frac{(1-\nu)^{-r}}{r(r+1)}\right), \quad && j > 0, \\ &= (1-\nu)^{-j}, && j < 0, \end{aligned}$$

provided $z_1 \geq \beta z_0$, where $\beta = 1 + \nu \log(1 - 1/\nu)$, and $-1 < \nu < 0$ $(= \mu)$. However since $(1-\nu)^j$ diverges none of these can be bounded above by x_j. Thus all x given by (21) are invariant over S for $\{P(t)\}$.

5. Some further remarks

The previous example shows that equations (5) and by implication equations (6) may not have a non-trivial solution. We can, however, employ the Harris-Veech conditions [7, 21] for the existence of invariant measures for discrete-time Markov processes to test for the existence of non-trivial solutions to (5) and (6), at least in the irreducible case. Harris proved that for a discrete-time Markov chain taking values in a countable set of states indexed by $C=\{0,1,2,\ldots\}$ with an irreducible transition matrix $\boldsymbol{J}=[J_{jk}]$, a sufficient condition for the equations

$$\sum_{j\in C} v_j J_{jk}=v_k,\quad k\in C,$$

to possess a positive solution, $\boldsymbol{v}=(v_j, j\in C)$, is that there exists an infinite set of states, K, such that for all $i=0,1,2,\ldots$

$$\lim_{j\to\infty}\lim_{k\to\infty:\,k\in K} L_{ki}(j)/L_{ki}=0,$$

where $L_{ki}(j)$ $(k\geqslant j)$ is the probability that the chain, starting in k, will first reach i jumping from a state with index $\geqslant j$, and $L_{ki}=L_{ki}(0)$ is the probability of ever reaching i starting from k. Veech proved that this condition is also necessary for $\boldsymbol{v}$ to be invariant for $\boldsymbol{J}$. Returning to the continuous-time process let $C\subseteq S$ be any irreducible class and assume, for the moment, that μ is strictly less than any of the $\{q_j\}$ for $j\in C$. Let $\boldsymbol{x}$ be any μ-subinvariant vector on C for $\{\boldsymbol{P}(t)\}$; there exists at least one by Theorem 3 of [13]. Now define $\bar{\boldsymbol{Q}}$ by (14) and observe that if $\boldsymbol{m}$ is μ-invariant for $\boldsymbol{Q}$, then $\boldsymbol{u}=(m_jx_j, j\in C)$ is *invariant* for $\bar{\boldsymbol{Q}}$. Thus $\boldsymbol{v}=(v_j, j\in C)$, where $v_j=u_j\bar{q}_j=m_jx_j(q_j-\mu)$, is invariant for the transition matrix, $\bar{\boldsymbol{J}}$, of the jump-chain corresponding to $\bar{\boldsymbol{Q}}$ which has elements

$$\bar{J}_{jk}=(1-\delta_{jk})q_{jk}x_k/((q_j-\mu)x_j),\quad j,k\in C.$$

Conversely, if $\boldsymbol{v}$ is invariant for $\bar{\boldsymbol{J}}$ (for some $\boldsymbol{x}$) then $\boldsymbol{u}=(v_j/(q_j-\mu), j\in C)$ is invariant for $\bar{\boldsymbol{Q}}$ and so $\boldsymbol{m}=(m_j, j\in C)$, where $m_j=u_jx_j^{-1}=v_j/((q_j-\mu)x_j)$, is μ-invariant for $\boldsymbol{Q}$. Thus we can apply the Harris-Veech condition to $\bar{\boldsymbol{J}}$ in order to determine whether or not μ-invariant measures exist for $\boldsymbol{Q}$. The probabilities $L_{ki}(j)$ can be expressed as

$$L_{ki}(j)=\sum_{r=j}^{\infty}\sum_{n=1}^{\infty}{}_i\bar{J}_{kr}^{(n)}\bar{J}_{ri}+\bar{J}_{ki}$$

where ${}_i\bar{J}_{kr}^{(n)}$ is the "taboo" probability that the jump-chain reaches r after n jumps starting in k without visiting i, a quantity that can be readily expressed in terms of $\bar{\boldsymbol{J}}$. The argument relating to μ-invariant vectors is similar. We choose any μ-subinvariant measure, $\boldsymbol{m}$, for $\{\boldsymbol{P}(t)\}$, define $\boldsymbol{Q}^*$ as in the proof of Theorem 2 and use the fact that $\boldsymbol{x}$ is μ-invariant for $\boldsymbol{Q}$ if and only if $\boldsymbol{v}=(m_jx_j, j\in C)$ is invariant for $\boldsymbol{Q}^*$ and hence for $\boldsymbol{J}^*$, the transition matrix with elements

$$J_{jk}^*=(1-\delta_{jk})m_kq_{kj}/((q_j-\mu)m_j),\quad j,k\in C.$$

The Harris–Veech conditions can then be applied to $\boldsymbol{J}^*$.

The assumption that μ be smaller than any of the $\{q_j\}$ for j in C is made to ensure that $\bar{q}_j$ (and q_j^*) be strictly positive. If this is not the case and $\mu=q_k$ for some $k\in C$ then by (7) we must have that $q_{jk}=0$ for all $j\neq k$ if μ-(sub)invariant measures are to exist. Conversely if R is the set of all such states (call them source states) then for μ-invariant measures to exist we must have that $\mu=q_k\leqslant\inf_{j\notin R}q_j$

for all $k \in R$. In Example 2 the source state, state 1, had $q_1 = 1$ but, although $q_j = 1$ for all $j \in S$, the only allowable value for μ was zero and so, in accordance with our findings, there could be no invariant measures. Observe that if S contains an absorbing state, k, μ must be zero and $q_{jk} = 0$ for all $j \neq k$. Thus, bar this exceptional case, there can be no invariant measures over S if S contains an absorbing state. The analogous implications of $\mu = q_k$ for μ-invariant vectors are less tacky. By a similar argument μ-(sub)invariant vectors can exist for $\boldsymbol{Q}$ only if $\mu = 0$ and state k is absorbing. To illustrate these points and further demonstrate Theorems 2 and 3 consider the simple random walk on the non-negative integers with an absorbing barrier at the origin.

Example 3. Define $\boldsymbol{Q}$ by setting $q_{00} = 0$ and for $j > 0$

$$\begin{aligned} q_{jk} &= p, && k = j+1, \\ &= -(p+q), && k = j, \\ &= q, && k = j-1, \\ &= 0, && \text{otherwise,} \end{aligned}$$

where both p and q are positive. There are clearly two irreducible classes, $\{0\}$ and $C = \{1, 2, \ldots\}$. Since 0 is an absorbing state (accessible from C) there can be no μ-invariant measures on S and μ-invariant vectors exist only if $\mu = 0$. It is easily checked that $\boldsymbol{x}$, given by $x_j = 1 + aj$, $j = 0, 1, 2, \ldots$, where $a \geq 0$, is invariant for $\{\boldsymbol{P}(t)\}$ over S.

Let us now turn our attention to the transient class C. By direct calculation of $P_{jk}(t)$ Seneta [19] determined the decay parameter, λ, together with all λ-invariant measures and vectors for C, and provided for them a quasistationary interpretation. Theorems 2 and 3 enable us to do this directly from the q-matrix.

The equations (5) for $\boldsymbol{m}$ can be written

$$\rho(1 - \delta_{j1}) m_{j-1} - (1 + \rho - \mu/q) m_j + m_{j+1} = 0, \quad j = 1, 2, \ldots, \tag{22}$$

where $\rho = p/q$. The equations (6) have a very similar form and on writing them down it is easy to see that their solution, $\boldsymbol{x}$, can be related to $\boldsymbol{m}$ by

$$x_j = \rho^{1-j} m_j. \tag{23}$$

Clearly solutions of (8) and (9) bear the same relationship. Therefore $y_j/m_j = z_j/x_j$ for each j and so if (22) admits a non-trivial solution, $\boldsymbol{m}$, then $\boldsymbol{m}$ is μ-invariant for $\{\boldsymbol{P}(t)\}$ if and only if $\boldsymbol{x}$, given by (23), is μ-invariant as well. Equation (22) is a homogeneous linear difference equation and so for a positive solution it is necessary that the characteristic equation

$$\alpha^2 - (1 + \rho - \mu/q)\alpha + \rho = 0 \tag{24}$$

has real roots, α_1 and α_2. The solution will be of the form

$$m_j = m_1 j (\sqrt{\rho})^{j-1}$$

if these roots are the same or

$$m_j = m_1\{(1+\rho-\mu/q)^2-4\rho\}^{-1/2}\{\alpha_1^j-\alpha_2^j\}$$

if they are distinct. Clearly (24) has real roots if and only if

$$|1+\rho-\mu/q| \geq 2\sqrt{\rho}$$

and these will be the same if $\mu = q(1+\rho \pm 2\sqrt{\rho}) = p+q \pm 2\sqrt{pq}$. However, if $\mu \geq p+q+2\sqrt{pq}$ some of the $\{m_j\}$ will be negative. For a non-trivial non-negative solution it is therefore necessary that $0 \leq \mu \leq \lambda = p+q-2\sqrt{pq}$, and so, by Theorem 4 of [23], λ is the decay parameter of the class C. It takes the value zero if and only if $p=q$. Thus if $p \neq q$ then C is *geometrically* transient in that for all j and k in C, $P_{jk}(t)$ converges to zero geometrically fast [13]. In fact, Seneta [19] shows that

$$P_{11}(t)\,\mathrm{e}^{\lambda t} = \mathrm{O}(t^{-3/2})$$

and this tells us that C is λ-transient; it is usually only possible to detect λ-positivity or otherwise from the q-matrix (see [17]). We distinguish the two cases (i) $\mu = \lambda \geq 0$ and (ii) $0 \leq \mu < \lambda$. The first has been dealt with by Seneta. The λ-invariant measure, $\boldsymbol{m}$, and vector, $\boldsymbol{x}$, are invariant for $\{\boldsymbol{P}(t)\}$ and are essentially unique, although this is not generally true in the λ-transient case. They are given by

$$m_j = j\rho^{j/2}, \quad j \in C,$$

$$x_j = j\rho^{-j/2}, \quad j \in C,$$

and $\boldsymbol{m}$ admits a quasistationary interpretation. For example, if $p<q$ $(\rho<1)$ the limit as t tends to infinity of

$$P\{X(t)=j \mid X(0)=i,\ X(t)\in C\}$$

exists and equals

$$m_j / \sum_{k\in C} m_k = (1-\rho^{1/2})^2 j\rho^{(j-1)/2}$$

for all j in C, independently of the initial state, i. In case (ii) the essentially unique μ-invariant measure and vector for $\boldsymbol{Q}$ are given by

$$m_j = \alpha_1^j - \alpha_2^j, \quad j \in C,$$

$$x_j = (\alpha_1/\rho)^j - (\alpha_2/\rho)^j, \quad j \in C,$$

where α_1 and α_2 are the (distinct) roots of (24), α_1 being the larger of the two. Now let us consider equations (8). Since $\nu < \mu < \lambda$ the essentially unique non-trivial, non-negative solution is given by

$$y_j = \beta_1^j - \beta_2^j$$

where $\beta_1 > \beta_2$ satisfy

$$\beta^2 - (1+\rho-\nu/q)\beta + \rho = 0.$$

But $1+\rho-\nu/q>1+\rho-\mu/q$ implies that $\beta_1>\alpha_1$ and so

$$\xi_j=\frac{y_j}{m_j}=\left(\frac{\beta_1}{\alpha_1}\right)^j\frac{(1-(\beta_2/\beta_1)^j)}{(1-(\alpha_2/\alpha_1)^j)}$$

diverges as $j\to\infty$. This implies that there is no non-trivial non-negative solution to (8) that can be bounded above by $\boldsymbol{m}$. Thus $\boldsymbol{m}$, and by implication $\boldsymbol{x}$, are μ-invariant on C for $\{\boldsymbol{P}(t)\}$. Observe that although $\boldsymbol{Q}$ is clearly regular Corollary 1 is of little use in this case. We have shown that *all* quantities that are μ-invariant for $\boldsymbol{Q}$ are also μ-invariant for $\{\boldsymbol{P}(t)\}$. However, we cannot guarantee that the sufficient conditions of Corollary 1 are satisfied for all values of p and q. For example, if $p\neq q$ $(\rho\neq 1)$ and $\mu=0$ then $m_j=|1-\rho^j|$ and $x_j=|1-\rho^{-j}|$ for $j=1,2,\ldots$. Thus $\boldsymbol{x}$ is bounded only when $\rho>1$. Of course Σm_j always diverges, as we should expect, since C is transient.

As a final remark it is of course possible to define invariant measures and vectors on any subset of S, in particular the union, U, of a collection of irreducible subclasses. In Example 1, for instance, if $U_n=\{n,n+1,\ldots\}$ and $\boldsymbol{m}$ and $\boldsymbol{x}$ are the essentially unique μ-invariant measures on $\mathbb{Z}$ for $\boldsymbol{Q}$ then $\boldsymbol{m}_n=(m_j, j\in U_n)$ and $\boldsymbol{x}_n=(x_j, j\in U_n)$ are clearly μ-invariant for $\boldsymbol{Q}$ over U_n. The proofs of Theorems 2 and 3, and relevant remarks relating to these, rest crucially on establishing the identities (13) and (18). If we wish to extend these results to statements about invariance over U we need to ensure that there is no contribution to the sum in (12) for states, i, outside U. Thus we need to assume that for each pair of classes (C, C') in U such that $C<C'$ there is *no* class, C'', outside U such that $C<C''<C'$. Put in another way, all paths in the state space leading from $j\in C$ to $k\in C'$ must be contained in U. This condition is somewhat reminiscent of the overtaking condition in networks of queues [24, 10]. It is clearly satisfied in the above example by the set U_n, and so, for all $n\in\mathbb{Z}$, $\boldsymbol{m}_n$ and $\boldsymbol{x}_n$ are invariant for $\{\boldsymbol{P}(t)\}$.

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